JOURNAL OF COMPUTATIONAL PHYSICS 34, 330–347 (1980)

Multigroup Representation of Fusion Product Orbits in a Plasma Column*

H. J. WILLENBERG

Mathematical Sciences Northwest, Inc., P.O. Box 1887, Bellevue, Washington 98009

Received October 6, 1978; revised January 19, 1979

A method is derived for describing the time-dependent behavior of α particles produced in a radially nonuniform slender plasma column as a distribution function among the possible orbits. A multigroup numerical approximation is introduced to analyze the development of the distribution function and its moments. Results are presented of calculations of the time-dependent α-particle energy spectrum and radial density, energy, and electron heating profiles in plasma columns with radii comparable to the α Larmor radius. This technique allows calculation of the α particle history at much more rapid rates than allowed by Monte Carlo techniques: The characteristic time scale is the α-electron slowing-down time rather than the cyclotron period.

1. INTRODUCTION

In order to understand the physics of a controlled thermonuclear plasma during fusion burn, it is essential to know the behavior of the reaction products. The reaction products of a deuterium tritium plasma include high-energy α particles which, in a magnetically confined plasma, travel in orbital trajectories with gyro-radii on the order of several centimeters for most magnetic fields of interest to thermonuclear plasma confinement. In understanding the interaction between these α particles and the reacting plasma, it is desirable to be able to describe the particle density, the thermalization rate, the plasma heating distribution, and the α energy spectrum. If the gyro-radius is not negligible compared to the plasma dimensions, or if the energy spectrum is not Maxwellian, a fluid-type description of the α particles is inadequate in defining these variables. In this case it is necessary to describe the α particles in terms of a population distribution function over the possible orbits.

A few α-particle trajectories typical of those in a slender, high-β solenoid are shown in Fig. 1. The magnetic field is parallel to the plasma column axis. Alpha particles travel through the plasma with large radii because of the low internal magnetic field in the high-β plasma. The orbital radii in the vacuum are smaller, causing the particles to reenter the plasma after an excursion into the vacuum.

The importance of the α particles on the fluid properties of the plasma has long

* Work performed under the auspices of the National Science Foundation and the U.S. Energy Research and Development Administration while the author was at the University of Washington.

0021-9991/80/030330-18$02.00/0

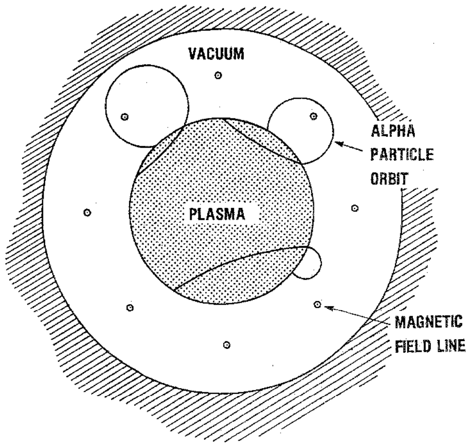


FIG. 1. Alpha-particle trajectories in a high-beta slender solenoid plasma column.

been known to researchers concerned with sustaining controlled thermonuclear reactions. Rose and Clark [1] showed that the presence of fusion-produced α particles can have a large effect on the pressure and energy balance of a cylindrical plasma. Sigmar and Joyce [2] demonstrated that a substantial fraction of the α-particle energy can be transferred to plasma ions in the 4-keV temperature range. Both of these analyses treated the plasma as a homogeneous fluid, although the latter did include dielectric effects. The neoclassical theory of α-particle transport has been developed for axisymmetric toroidal plasmas assuming the α particles have a Maxwellian energy distribution [3, 4]. Düchs and Pfirsch extended this theory by numerically solving an energy-space Fokker–Planck equation, and representing the quasi-steady solution as a linear combination of three Maxwellian distributions [5]. Tsuji *et al.* [6] solved an energy-dependent Fokker–Planck equation with finite particle confinement times to determines the time-dependent energy spectrum in a uniform plasma. Corman *et al.* [7] have applied multigroup diffusion techniques to a Fokker–Planck equation to investigate slowing down and spatial transport of α particles in a field-free plasma. Finite-sized α orbits were not considered in any of these analyses.

Stringer [8] and McAlees [9] determined the phase-space distribution function of α particles of constant energy in a tokamak plasma with radial structure. These reports treated orbital motion outside the context of neoclassical diffusion theory for the first time, but did not investigate the time dependence of the α-particle distribution. Petrie and Miley [10] investigated α orbit drift motion during slowing down in a toroidal plasma. Their technique approximates the change in individual α-particle orbits while slowing down and uses a multigroup description of the particles based on their orbital characteristics. Adherence to guiding center theory is assumed. Steinhauer [11] treated the problem of time-dependent slowing down in the case where α-particle orbits make excursions outside the plasma by defining an energy-dependent residence factor by which to increase the slowing-down time above that in a uniform plasma.

With the exception of Ref. [10], it appears that no investigation prior to this one has been made of time-dependent α-particle behavior in a nonuniform, magnetized plasma. The effects of large orbits on the phenonomena of α slowing down and associated plasma heating have not been explored for a distributed energy spectrum in an inhomogeneous plasma. A method is presented here to follow the development of the α particles produced by fusion in a slender, cylindrical, magnetically confined plasma column which includes a fusion source and nonuniform thermalization. This method allows determination of the radial properties of the α behavior, including density, pressure, and heating rate. An application is described here of time-dependent α-particle behavior in a fixed background plasma.

Although the application described here is not self-consistent in order to focus on the mathematical technique, the method also allows determination of α-particle and background plasma behavior in a fully self-consistent manner, with an appropriate description of the plasma physics and the α-plasma interaction. Such an application has been performed, where the α particles transfer energy to the plasma electrons and ions in a nonuniform manner during thermalization, and the plasma is heated and expands [12][1]. The results of this application have been published in another article [13].

In this paper we will describe the α-particle behavior in terms of a distribution function over the orbital parameters in axisymmetric, cylindrical geometry. The mathematical formulation of the distribution function development is presented in Section 2. The numerical techniques used to solve for the time development of the distribution function are presented in Section 3. The techniques for converting the distribution to space-dependent fluid variables are described in Section 4. The results of applying this methodology to two particular geometries are shown in Section 5.

2. α-Particle Distribution Function

The geometry for which the methodology described here was formulated is that of an infinitely long, axisymmetric plasma column of radius comparable to an α-particle Larmor radius at birth, confined by an axial magnetic field. This description applies to the linear θ pinch and the laser- and e-beam-heated solenoid plasmas. All spatial variables are assumed to be axially and azimuthally symmetric. The plasma column is nonuniform in the radial direction.

Neglecting electrostatic fields, the α-particle Lagrangian in a magnetic field is

$$L = 1/2m\mathbf{u} \cdot \mathbf{u} + q\mathbf{A} \cdot \mathbf{u},$$

where $\mathbf{A} = A(r)\, \hat{e}_\theta$ is the vector potential, defined by

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

[1] For a verification of the assumptions made in the derivations and a comparison with other computational results, the reader is referred to Ref. [12].

A single-particle Hamiltonian is derived from this Lagrangian in terms of the canonical momenta

$$p_r = mu_r, \tag{1a}$$

$$p_\theta = mru_\theta + qrA, \tag{1b}$$

and

$$p_z = mu_z \tag{1c}$$

as

$$H = \frac{p_r^2}{2m} + \frac{p_z^2}{2m} + \frac{[p_\theta - qrA(r)]^2}{2mr^2}. \tag{2}$$

Since r is the only spatial coordinate, the canonical angular and axial momenta p_θ and p_z are constants of the motion. Furthermore, on a time scale for which A is static, the Hamiltonian does not explicitly depend on time, and is also a constant of the motion. With the axial symmetry of this problem, these three adiabatic invariants, H, $P \equiv p_\theta$, and $v \equiv p_z/m$ uniquely determine an α-particle trajectory. The velocity components can be obtained as a function of r by inversion of Eq. (1).

$$u_z = v, \tag{3a}$$

$$u_\theta = \frac{P}{mr} - \frac{qA}{m}, \tag{3b}$$

$$u_r = \pm\left[\frac{2H}{m} - \left(\frac{P}{mr} - \frac{qA}{m}\right)^2 - v^2\right]^{1/2}. \tag{3c}$$

The collisional Boltzmann equation for a collection of particles in a magnetic field is

$$\frac{\partial f}{\partial t} + \mathbf{u}\cdot\nabla f + \frac{q}{m}(\mathbf{u}\times\mathbf{B})\cdot\nabla_{\mathbf{u}} f = \left.\frac{\partial f}{\partial t}\right|_{\text{collisions}} + \left.\frac{\partial f}{\partial t}\right|_{\text{source}}. \tag{4}$$

Any function f of the constants of the motion is a solution to the time-independent Vlasov equation [14], i.e.,

$$\mathbf{u}\cdot\nabla f + \frac{q}{m}(\mathbf{u}\times\mathbf{B})\cdot\nabla_{\mathbf{u}} f = 0$$

if f can be expressed explicitly in terms of the constants of the motion. Therefore, if we describe the α-particle distribution in terms of the adiabatic invariants H, P, and v, then the Boltzmann equation reduces to

$$\frac{\partial f(H, P, v)}{\partial t} = \left.\frac{\partial f}{\partial t}\right|_{\text{collisions}} + \left.\frac{\partial f}{\partial t}\right|_{\text{source}} \tag{5}$$

on a time scale for which the adiabatic invariants do not change rapidly for a single particle. This equation yields the time-dependence of the distribution of α particles among the various orbits, as the particles are produced and slow down. Since the single-particle trajectories are the characteristic curves of the Boltzmann equation [15], the terms on the right-hand side of Eq. (5) must be determined along a trajectory, i.e., $\partial f/\partial t$ must be determined for fixed H, P, and v.

This result allows us to analyze the development in time of a collection of α particles. We can describe the distribution of particles by the number of α particles which follow each orbit. We label each α orbit by its identifying adiabatic invariants, and define a distribution function $f(H, P, v)$ such that $f(H, P, v)\, dHdPdv$ is the number of α particles which follow orbits within dH of H, dP of P, and dv of v. This form of distribution function is fully equivalent to the more common phase-space distribution function $f(r, \mathbf{u})$, and has the great advantage of already being explicitly expressed in terms of the Boltzmann equation characteristics. Along these characteristics,

$$\frac{df}{dt} = \frac{\partial f}{\partial H}\frac{\partial H}{\partial t} + \frac{\partial f}{\partial P}\frac{\partial P}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial f}{\partial t}. \tag{6}$$

For any α particle, the changes in H, P, and v are negligible on a time scale comparable to a cyclotron period, and the integrity of a single-particle orbit is preserved on this time scale. On a longer time scale, H, P, and v change slowly as a result of α-particle interactions with the background plasma.

Equation (6) is a first-order equation in time which can be solved for the time-dependent α-particle distribution function $f(H, P, v; t)$ by a perturbation method, provided that the rate at which particles change H, P, and v is slow compared to an orbital period. Given the value of f at any time, $\partial f/\partial H$, $\partial f/\partial P$, and $\partial f/\partial v$ can be simply evaluated for each particular orbit (H, P, v). The partial time derivatives for each orbit must be evaluated by a suitable method of averaging the rate of change of the orbital parameters over many orbital periods. With such an average value obtained,

$$\frac{df}{dt} = \frac{\partial f}{\partial H}\left\langle\frac{\partial H}{\partial t}\right\rangle + \frac{\partial f}{\partial P}\left\langle\frac{\partial P}{\partial t}\right\rangle + \frac{\partial f}{\partial v}\left\langle\frac{\partial v}{\partial t}\right\rangle + \left\langle\frac{\partial f}{\partial t}\right\rangle, \tag{7}$$

where the brackets imply average values, taken along the characteristic curves, i.e., along the particle trajectories. These averages are taken by weighting the values at each radial location by the time $|(dr/u_r)(H, P, v)|$ that a particle following the trajectory (H, P, v) spends at the location. For each parameter g to be averaged along a particular orbit (H, P, v),

$$\langle g\rangle = \int_{r_{\min}(H,P,v)}^{r_{\max}(H,P,v)} g(r; H, P, v)\,\frac{dr}{u_r(r; H, P\ v)}\Big/\left[\int_{r_{\min}(H,P,v)}^{r_{\max}(H,P,v)} \frac{dr}{u_r(r; H, P, v)}\right], \tag{8}$$

where the limits of integration are the radial turning points of the orbit represented by (H, P, v).

The α particles lose their energy as a result of Coulomb interactions with the background plasma. The rate of energy loss due to the electrons and ions is a function of plasma conditions. Given $\partial E/\partial t$, the time derivatives can be quickly evaluated if the changes are assumed to be the cumulative result of small-angle scattering, which is valid for α-electron interactions through the entire energy range, and a good approximation for α-ion interactions at all energies but the low energy end of the α spectrum. Large-angle deflections do not represent a major effect. In dimensionless units for which the plasma radius, α-particle birth energy and speed, plasma radius, vacuum magnetic field, and maximum angular momentum at the plasma edge are unity,

$$\frac{\partial H}{\partial t} = \frac{\partial E}{\partial t}, \tag{9a}$$

$$\frac{\partial P}{\partial t} = \frac{\partial E}{\partial t} \frac{P - rA(r)}{2H} - rE_\theta(r), \tag{9b}$$

and

$$\frac{\partial v}{\partial t} = \frac{\partial E}{\partial t} \frac{v}{2H}. \tag{9c}$$

A is expressed in Coulomb gauge, where the azimuthal electric field is a result of the time dependence of A. The source term $\partial f/\partial t\,|_{\text{source}}$ is assumed to be isotropic and proportional to $n^2 \langle \sigma v \rangle$ for orbits such that $H = 1$, and zero for $H < 1$. Dimensionless units as defined here are used throughout this article except when specifically labeled in SI units.

Several authors [16–19] have developed theories to describe the rate of energy loss of α particles slowing down in a uniform plasma. Kammash and Galbraith developed an analytical expression for the energy loss rate which applies to slowing down in the classical and quantum regimes and varies smoothly between the two for intermediate energies [17]. This expression is given in the reference in Gaussian units as

$$\begin{aligned} \frac{dE}{dt} = -\frac{4\pi n e^4}{\mu v} \Big\{ & \Big[\frac{8}{\pi^{1/2}} R e^{-R^2} - \frac{4 m_\alpha}{m_\alpha + M} \phi(R)\Big] \ln \beta \\ & + \Big[\frac{m_\alpha}{m_\alpha + M} \{\phi(R + \alpha^{1/2}) + \phi(R - \alpha^{1/2})\} \\ & + \frac{e^{-(R+\alpha^{1/2})^2} - e^{-(R-\alpha^{1/2})^2}}{(\pi\alpha)^{1/2}}\Big] \ln \alpha \Big\}, \end{aligned} \tag{10}$$

where

$\phi(x)$ is the standard error function,

M is the field particle mass, m_e or m_i,

μ is the reduced mass, $Mm_\alpha/(M + m_\alpha)$

$$R \equiv \left(\frac{M}{m_\alpha}\right)^{1/2} \left(\frac{E}{kT}\right)^{1/2},$$

$$\alpha \equiv \frac{M}{2kT} \left(\frac{4e^2}{\hbar}\right)^2,$$

$$\beta \equiv \frac{Me^2}{\mu kT} \left(\frac{8\pi n e^2}{kT}\right)^{1/2}.$$

This expression is used throughout the present investigation for the α energy loss rate. Anomalous slowing down, such as that considered by Sigmar and Chan [21], Belikov *et al.* [22], or Tsytovich [23], is not considered here, but is treated parametrically in Ref. [13].

The study of the α-particle evolution in time has been reduced to determination of $\partial H/\partial t$, $\partial P/\partial t$, and $\partial v/\partial t$ as a result of slowing down, and $\partial f/\partial t$ due to the fusion source. In a small time increment (microsecond scale) the change in population following a given orbit (H, P, v) can be found by evaluating the averages $\langle \partial H/\partial t\rangle$, $\langle \partial P/\partial t\rangle$, $\langle \partial v/\partial t\rangle$, and $\langle \partial f/\partial t\rangle$ along that orbit and factoring by the appropriate distribution derivatives $\partial f/\partial H$, $\partial f/\partial P$, and $\partial f/\partial v$. With this result, the change in $f(H, P, v)$ in time Δt is

$$f(H, P, v; t + \Delta t) - f(H, P, v; t) = \frac{df}{dt}\Delta t. \tag{11}$$

This technique is similar to that of Petrie and Miley [10] since it describes α particle orbits with a multigroup approximation, but differs from that technique in describing the shifts in distribution of particles along fixed orbits rather than following the shift in orbit of a collection of pseudoparticles, and in its applicability when the guiding-center approximation breaks down.

3. Numerical Technique

In order to find the time development of the entire distribution function, the range of H, P, and v is divided into groups of finite width ΔH_j, ΔP_j, and Δv_k. The value of the function $f(H_i, P_j, v_k)$, abbreviated f_{ijk}, is a measure of the population of α particles following trajectories described by H, P, and v in the domain $H_i - \Delta H_i/2 < H \leqslant H_i + \Delta H_i/2$, $P_j - \Delta P_j/2 < P \leqslant P_j + \Delta P_j/2$, $v_k - \Delta v_k/2 < v \leqslant v_k + \Delta v_k/2$. The change in f_{jik} in a time increment Δt can be found for each possible combination of (ijk) by the procedure of Eqs. (7)–(11), provided that this change is small compared to f_{ijk}. All the necessary orbital parameters are evaluated assuming that H_i, P_j, and v_k are the approximate values for all f_{ijk} particles within the (ijk) group.

As the α particles are produced and slow down in the background plasma, the distribution function changes. For time scales which are short compared to the slowing down time, it may be assumed that, as a result of small-angle scattering, the α particles are transferred into neighboring orbits, for which the identifying indices do not change by more than 1; i.e., the change in H for a given particle is not more than ΔH in a single timestep, the change in P not more than ΔP, etc. Let us investigate the change in population distribution involving a particular orbit (ijk). At time t, f_{ijk} α particles travel in this orbit. The radial velocity of particles in this orbit is a known function of r. Dividing the radial range into intervals identified by r_l,

$$u_{r_{ijk}}(r_l) = \left\{H_i - v_j^2 - \left[\frac{P_k}{r_l} - A(r_l)\right]^2\right\}^{1/2}. \tag{12}$$

The radial turning points $r_{l\min}$ and $r_{l\max}$ for this orbit are found by setting $u_{r_{ijk}}(r_l) = 0$ and solving for r_l:

$$\left[\frac{P_k}{r_l} - A(r_l)\right]^2 = H_i - v_j^2.$$

The fraction of time that a particle on orbit (ijk) spends in the interval Δr_l about r_l is

$$\frac{\Delta r_l}{u_{r_{ijk}}(r_l)} \Big/ \sum_{l=l_{\min}}^{l_{\max}} \frac{\Delta r_l}{u_{r_{ijk}}(r_l)}.$$

While the particle is in this region, the time rate of change of energy is given by $\partial H(H_i, r_l)/\partial t$, since $\partial H/\partial t$ is a function of H, n, T_e and T_i, while n, T_e, and T_i are functions of r. Therefore, the average energy change of particles on orbit (ijk) in interval Δr_l during time Δt is

$$\delta H = \left[\frac{\partial H}{\partial t}(H_i, r_l) \frac{\Delta r_l}{u_{r_{ijk}}(r_l)} \Big/ \sum_{l=l_{\min}}^{l_{\max}} \frac{\Delta r_l}{u_{r_{ijk}}(r_l)}\right] \Delta t. \tag{13}$$

This averaging process represents a multigroup equivalent to the process described by Eq. (8). The energy groups are labeled according to the ordering $H_1 = 1$, $H_{i+1} < H_i < H_{i-1} < \cdots < H_2 < H_1$.

The assumption is made that the number of particles losing enough energy to drop into the next energy level $i + 1$ is proportional to the population of level i, times the ratio of the average energy change to the spacing, i.e., the number of particles leaving orbit (ijk) and entering orbit $(i + 1, j, k)$ while in the interval r_l is

$$\Delta f_{i+1,j,k} = f_{ijk} \frac{\delta H}{\Delta H_i}, \tag{14}$$

where δH is defined in Eq. (13) and ΔH_i is the width of energy interval i. As the particles enter orbit $(i + 1, j, k)$, they leave orbit (i, j, k):

$$\Delta f_{ijk} = -\Delta f_{i+1,j,k}. \tag{15}$$

The same procedure is performed for particles changing momentum in radial interval r_l. In the same manner as Eq. (13) was derived,

$$\delta P = \left[\frac{\partial P}{\partial t}(H_i, v_j, P_k, r_l) \frac{\Delta r_l}{u_{r_{ijk}}(r_l)} \Big/ \sum_{l=l_{\min}}^{l_{\max}} \frac{\Delta r_l}{u_{r_{ijk}}(r_l)}\right] \Delta t, \tag{16}$$

where

$$\frac{\partial P}{\partial t} = \frac{\partial H}{\partial t}(H_i, r_l) \frac{P_k - r_l A(r_l)}{2H_i} - r_l E(r_l),$$

and

$$\delta v = \left[\frac{\partial v}{\partial t}(H_i, v_j, r_l) \frac{\Delta r_l}{u_{r_{ijk}}(r_l)} \Big/ \sum_{l=l_{\min}}^{l_{\max}} \frac{\Delta r_l}{u_{r_{ijk}}(r_l)}\right] \Delta t, \tag{17}$$

where

$$\frac{\partial v}{\partial t} = \frac{\partial H}{\partial t} (H_i , r_j) \frac{v_j}{2H_i} .$$

Finally, the number of α particles produced by fusion per unit volume per unit time in radial interval Δr_l is

$$C(T_{i_l}^{\text{new}})^3 (n_l^{\text{new}})^2,$$

where C is a normalization parameter representing the fusion rate at unit n and T_i. Here the fusion cross-section term $\langle \sigma v \rangle$ has been approximated as

$$\langle \sigma v \rangle = 1.17 \times 10^{-25} T^3 \text{ m}^3/\text{sec},$$

where T is measured in keV. Considering the source to be isotropic,

$$\Delta f_{ijk} |_{\text{source}} = \nu_{ijk} = \frac{2 \Delta v_j \, \Delta P_k \, C(T_i^{\text{new}})^3 \, (n_l^{\text{new}})^2 \, \Delta r_l \, \Delta t}{\pi H_i^{1/2}} \qquad \text{for} \quad i = 1. \tag{18}$$

It is now possible to determine $f(H, P, v; t + \Delta t)$, given $f(H, P, v; t)$. This is done by adding the changes in each radial interval throughout the plasma. For each interval Δr_l, the change in f due to variation of particles on orbit (ijk) is

$$\Delta f_{i,j,k} = -f_{ijk} \left(\left| \frac{\delta H}{\Delta H_i} \right| + \left| \frac{\delta v}{\Delta v_j} \right| + \left| \frac{\delta P}{\Delta P_k} \right| \right) + \begin{cases} \nu_{ijk} & \text{if} \quad i = 1, \\ 0 & \text{if} \quad i \neq 1, \end{cases} \tag{19a}$$

$$\Delta f_{i+1,j,k} = f_{ijk} \frac{\delta H}{\Delta H_i}, \tag{19b}$$

$$\Delta f_{i,j+1,k} = f_{ijk} \frac{\delta v}{\Delta v_j}, \tag{19c}$$

and

$$\Delta f_{i,j,k\pm 1} = \pm f_{ijk} \frac{\delta P}{\Delta P_k} \tag{19d}$$

with the right-hand side positive. These operations are performed at all radial intervals l, on all occupied orbits (ijk) where the radial velocity $u_{r_{ijk}}(r_l)$ is real. f_{ijk} is always given as an initial condition at any time, and the fractional rates of change $\delta H / \Delta H_i$, etc., are found through the use of Eqs. (13) through (19). After performing the operations for all allowable $(ijkl)$ at time t, a new distribution function is found at time $t + \Delta t$.

The multigroup approximation to Eq. (7) can now be seen when all the Δf's are added in the same expression. At interval Δr_l,

$$\begin{aligned} \Delta f_{ijk} = {} & f_{i-1,j,k} \frac{\delta H}{\Delta H_{i-1}} - f_{ijk} \frac{\delta H}{\Delta H_i} + f_{i,j-1,k} \frac{\delta v}{\Delta v_{j-1}} - f_{ijk} \frac{\delta v}{\Delta v_j} \\ & \mp f_{i,j,k\pm 1} \frac{\delta P}{\Delta P_{k\pm 1}} \pm f_{ijk} \frac{\delta P}{\Delta P_k} + \nu_{ijk} \delta_{1,i} , \end{aligned} \tag{20}$$

where $\delta_{1,i}$ is the Kronecker delta function. Adding the results over all radial intervals,

$$\begin{aligned}\Delta f_{ijk} = {} & \frac{f_{i-1,j,k}}{\Delta H_{i-1}}\left\langle\frac{\partial H}{\partial t}\Delta t\right\rangle_{i-1} - \frac{f_{ijk}}{\Delta H_i}\left\langle\frac{\partial H}{\partial t}\Delta t\right\rangle_i \\ & + \frac{f_{i,j-1,k}}{\Delta v_{j-1}}\left\langle\frac{\partial v}{\partial t}\Delta t\right\rangle_{j-1} - \frac{f_{ijk}}{\Delta v_j}\left\langle\frac{\partial v}{\partial t}\Delta t\right\rangle_j \\ & \pm \frac{f_{i,j,k\pm1}}{\Delta P_{k\pm1}}\left\langle\frac{\partial P}{\partial t}\Delta t\right\rangle_{k\pm1} \pm \frac{f_{ijk}}{\Delta P_k}\left\langle\frac{\partial P}{\partial t}\Delta t\right\rangle_k \\ & + \left\langle\frac{\partial f}{\partial t}\Delta t\right\rangle_{\text{source}}. \end{aligned} \tag{21}$$

Dividing by Δt yields the equation

$$\frac{\Delta f}{\Delta t} \approx \frac{\Delta f}{\Delta H}\left\langle\frac{\partial H}{\partial t}\right\rangle + \frac{\Delta f}{\Delta v}\left\langle\frac{\partial v}{\partial t}\right\rangle + \frac{\Delta f}{\Delta P}\left\langle\frac{\partial P}{\partial t}\right\rangle + \left\langle\frac{\partial f}{\partial t}\right\rangle. \tag{22}$$

4. Macroscopic Variables

In the previous sections, a method has been presented for determining the characteristics of the α-particle distribution function as the particles originate and slow down in a thermonuclear plasma column. The distribution function itself, however, yields relatively little information about particle behavior. Of much greater interest are the fluid-like properties of the α particles, such as perticle density, pressure, and heating rates. These properties are derived by taking moments of the distribution function.

Keeping in mind that f_{ijk} is the number of particles whose motion is described by the orbit (ijk), and that the fraction of time that a particle on this orbit spends in the radial interval Δr_l is

$$\frac{\Delta r_l}{u_{r_{ijk}}(r_l)} \Big/ \sum_{l=l_{\min}}^{l_{\max}} \frac{\Delta r_l}{u_{r_{ijk}}(r_l)},$$

the number of particles following orbit (ijk) present at a given time in the radial interval is

$$f_{ijk}\frac{\Delta r_l}{u_{r_{ijk}}(r_l)} \Big/ \sum_{l=l_{\min}}^{l_{\max}} \frac{\Delta r_l}{u_{r_{ijk}}(r_l)}.$$

Considering all particles passing through the interval, the total number of particles present in radial interval Δr_l at a given time is

$$\sum_{ijk} f_{ijk}\left(\frac{\Delta r_l}{u_{r_{ijk}}(r_l)} \Big/ \sum_{l=l_{\min}}^{l_{\max}} \frac{\Delta r_l}{u_{r_{ijk}}(r_l)}\right).$$

The quantity to be summed over is a finite difference form of the average f, as given by Eq. (8), so the number of particles present in radial interval Δr_l is

$$\sum_{ijk} \langle f \rangle_{ijk} .$$

The volume of the radial interval l per unit length is $\pi(r_l^2 - r_{l-1}^2)$, so the number density of α particles in radial interval Δr_l is

$$n_\alpha(r_l) = \frac{\sum_{ijk} \langle f \rangle_{ijk}}{\pi(r_l^2 - r_{l-1}^2)} .$$

Nondimensionalizing f such that

$$f = 2\pi R^2 N f',$$

where R is a characteristic plasma column radius and N a characteristic number density

$$n_\alpha'(r_l') = \frac{2 \sum_{ijk} \langle f \rangle_{ijk}}{r_l'^2 - r_{l-1}'^2}, \tag{23}$$

where the primes denote nondimensional forms.

Other fluid variables can be found in a similar manner. For instance, the total heating rate due to all α particles is

$$-\sum_{ijk} \left\langle \frac{\partial H}{\partial t} (r_l)\Big|_e f \right\rangle_{ijk} .$$

The volumetric heating rate of electrons is then

$$Q_{\alpha e}(r_l) = \left[\sum_{ijk} \left\langle \frac{\partial H}{\partial t} (r_l)\Big|_e f \right\rangle_{ijk}\right] \Big/ \pi(r_l^2 - r_{l-1}^2). \tag{24}$$

5. Results

The methodology described above has been applied to plasma columns of solenoid geometry [20]. The geometry is shown in Fig. 2. In this approach to fusion, a long cylindrical plasma column is confined by an axial magnetic field. A slender plasma column is heated along the solenoid axis by an external source, either a laser or electron beam. In the fast solenoid version, the magnet current compresses the plasma, so that the region between the heated plasma and the first wall is vacuum. In the slow solenoid version, this region is filled with cooler plasma. Both cases are considered below. The assumption that H, P, and v change slowly on a cyclotron period time scale is valid as long as the slowing-down time and the deflection time are much longer than the orbital period. This is true in all cases considered here.

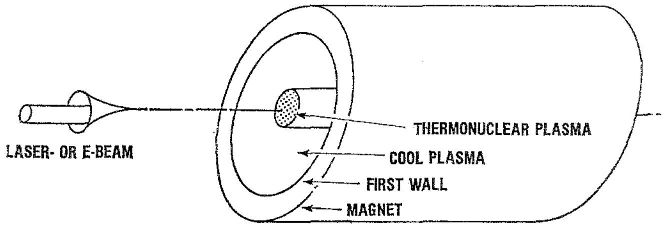
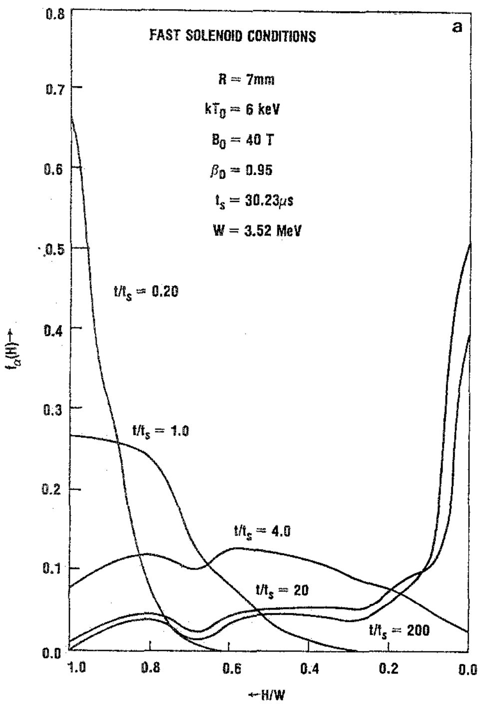


FIG. 2. Solenoid plasma geometry.

Fast Solenoid Case

In the fast solenoid case considered here, the plasma temperature is 6 keV, its radius is 7 mm, and $\beta = 0.95$. The plasma is uniform with a sharp boundary, surrounded by a vacuum with a magnetic field of 40 T. These conditions correspond to a plasma density of 3.15×10^{23} ions/m^3. An isotropic α-particle source appropriate to these plasma properties was introduced, and the evolution of the α-particle properties was followed, assuming that the plasma remains stationary. The α-particle energy loss rate published by Kammash and Galbraith [17] was assumed. The α-particle energy spectrum is presented in Fig. 3a, the particle density in Fig. 3b, average energy in Fig. 3c, and the α-electron heating rate in Fig. 3d.



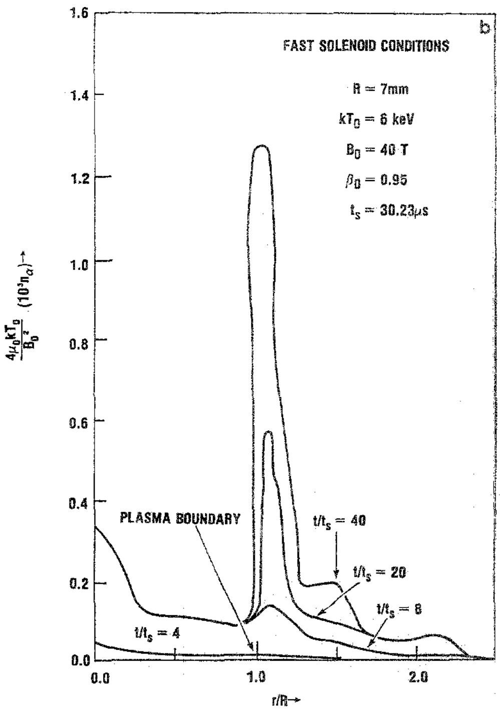
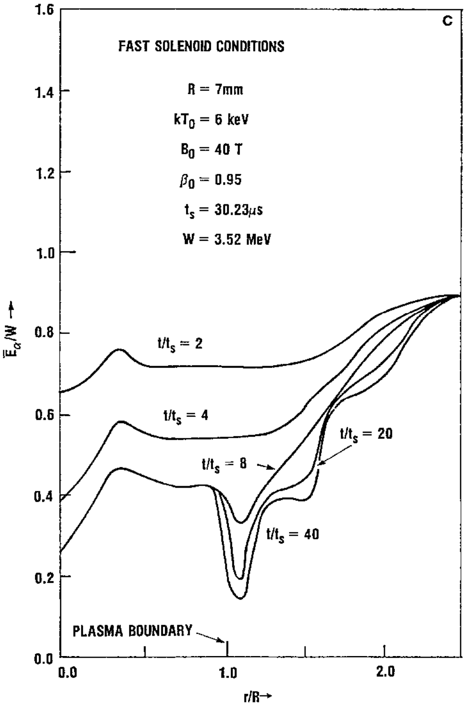


FIG. 3a. Normalized α spectrum, $f_\alpha(H)$, fast solenoid conditions in stationary plasma.
FIG. 3b. Alpha-particle density profile, fast solenoid conditions in stationary plasma.



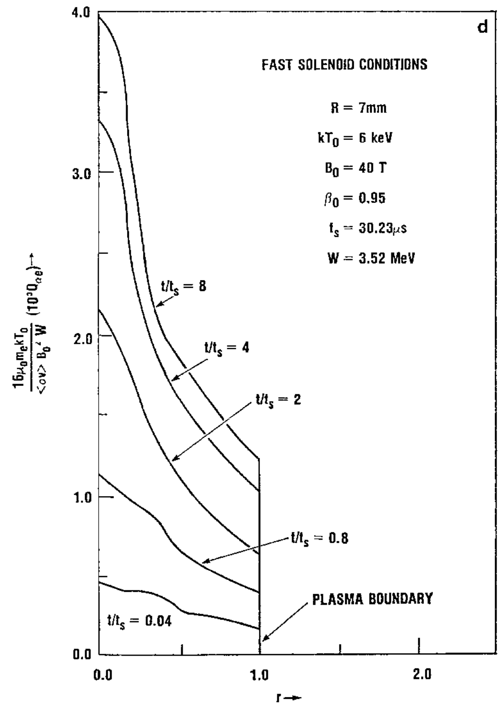


FIG. 3c. Mean α-particle energy, fast solenoid conditions in stationary plasma.
FIG. 3d. Electron heating profile, fast solenoid conditions in stationary plasma.

The α-electron slowing-down time for such a plasma is $t_s = 30\,\mu$sec—much less than a characteristic heating time. The α-particles are continuously born at kinetic energy $H = 1$. In this descriptive example the birthrate is constant as well, because the plasma properties remain fixed. The energy spectrum at various times is shown in Fig. 3a. The ordinate is the fraction of α-particles with kinetic energy within $H \pm 0.1$. For times smaller than any slowing-down time, the spectrum is predominantly high-energy. As time evolves and the α-particles slow down, the spectrum softens. The particles born first begin to reach the thermal energy zone at a time of about 75 μsec, $t/t_s = 2.5$. At this time, the spectrum is still skewed on the high-energy end, because particles are still being born, and those with a large residence time outside the plasma have not slowed much. By $t/t_s = 4$, the spectrum is flattening out and beginning to approach a quasi-steady state, where the number of particles slowing down through each energy region is equivalent to the number being born at $H = 1$. For later times ($t/t_s \geqslant 10$), the nonthermal α-spectrum is approximately stationary, with particles being introduced at $H = 1$ and lost at the same rate to the thermal energy group. After this time, the only change in the spectrum is that the thermal group grows at the birthrate.

In the same time that the nonthermal α spectrum takes to approach a steady state, the fluid properties of the α particles inside the plasma also approach a steady state.

By the time $t/t_S = 10$, the density profile inside the plasma is steady, as shown in Fig. 3b, as are the other fluid properties, such as the average energy profile shown in Fig. 3c, and the electron heating profile shown in Fig. 3d. The density profile peaks at the axis, and decays more or less uniformly to zero at 1 Larmor diameter outside the edge of the plasma, except for a very large peak at the very edge of the background plasma. This peak is a result of the high plasma β considered in this case. For very high β geometry, where the internal magnetic field is very low, the α-particle trajectories can be characterized by straight lines inside the plasma, and circles outside. High-energy α particles, with large Larmor radii, execute orbits extending far from the plasma edge. In the absence of an appreciable internal magnetic field, all α-particles reach the plasma edge, regardless of their energy. Once having reached the plasma edge, the excursion distance out of the plasma depends strongly on the perpendicular energy. Alpha particles of small kinetic energy do not penetrate deeply into the vacuum field. Those higher-energy particles with deep excursions out of the plasma slow down and pile up at the edge of the plasma. As the number of thermal α-particles increases, the number of particles in this class increases, resulting in the peak in number density just outside the plasma seen in Fig. 3b. This explanation of the peak is substantiated in Figs. 3c and 4. In Fig. 3c, it can be seen that the development of the density peak just outside the plasma coincides with the development of a depression in the mean α-particle energy, indicating a buildup of low-energy particles in that

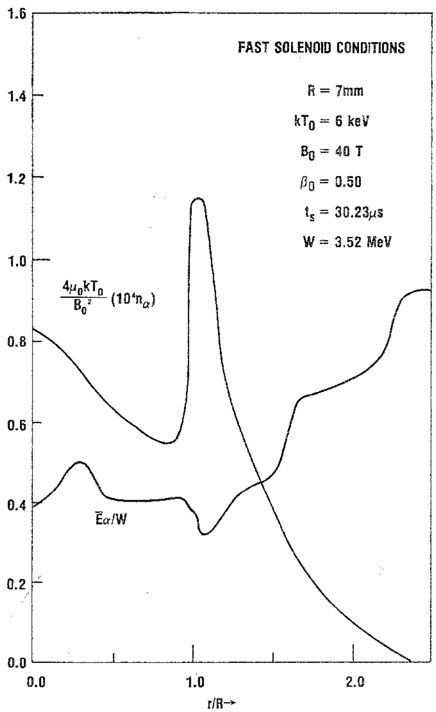


FIG. 4. Alpha profiles, $\beta = 0.5$, fast solenoid conditions in stationary plasma.

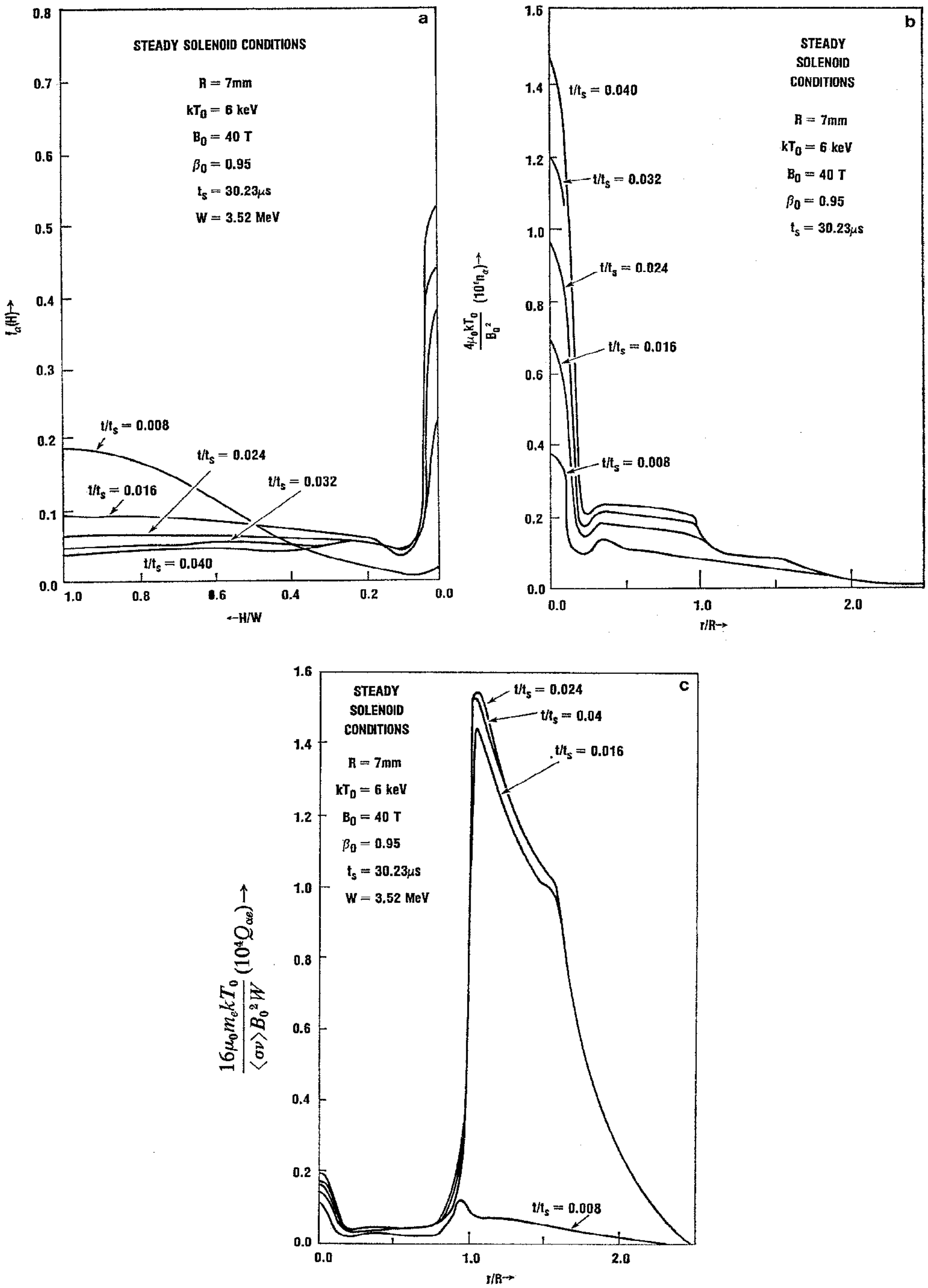


FIG. 5. (a) Normalized α spectrum, steady solenoid conditions in stationary plasma. (b) Alpha-particle density profile, steady solenoid conditions in stationary plasma. (c) Electron heating profile, steady solenoid conditions in stationary plasma.

region. Furthermore, the same profiles are shown in Fig. 4 for the case of lower β at a time when the average α-particle energy is the same. For $\beta = 0.5$, the ratio of peak density to centerline density at $t/t_s = 40$ is 1.4, compared to 1.7 for $\beta = 0.95$ at $t/t_s = 20$, indicating a milder peak for the same degree of thermalization. This strong peaking is reduced when the effect of α-particle heating on the plasma is considered in a self-consistent manner [13], because the plasma expands to encompass the slow α-particles.

Steady Solenoid Case

The rate of α-particle slowing down in a cold plasma is much more rapid than in a hot plasma. Considering the fact that, in addition, there is no vacuum region where the particles do not slow down, we see that thermalization is very rapid indeed. The methodology presented in Sections 2–4 above has also been applied to a stationary, steady solenoid plasma. In this case, $R = 7$ mm, $T_0 = 6$ keV, $B_0 = 40$ T, and $\beta = 0.95$ on the axis, just as in the fast solenoid case, but now the hot plasma is surrounded by a cold plasma with $T = 10$ eV, instead of a vacuum. A temperature profile of

$$\begin{aligned} T &= 1 - r^6 \qquad \text{for} \quad r \leqslant 1 \\ &= 0.00167 \qquad \text{for} \quad r > 1 \end{aligned}$$

was selected and B/n is constant everywhere. The α-particle energy spectrum is shown in Fig. 5a, the particle density in 5b, and the α-electron heating rate in Fig. 5c.

It can be easily seen from all these figures that quasi-steady-state conditions are reached very rapidly in the case of the steady solenoid. Within 1 μsec, the only part of the energy spectrum that is not steady is the thermal energy group, the only part of the density profile that is not steady is near the axis, and the electron heating profile is changing very slowly. It is interesting to note that, whereas in the fast solenoid case quasi-steady conditions were reached when the interior became steady and particles continued to pile up outside the plasma, in the solenoid case the external regions quickly approach steady state and particles continue to pile up inside the hot plasma. In Fig. 5c, it is seen that the large majority of the α energy is deposited in the electrons within a Larmor diameter outside the hot–cold plasma interface.

6. Conclusions

A new methodology has been developed to analyze the dynamics of the α particles produced by fusion in a linear, magnetically confined plasma column with radially nonuniform fluid properties. The thermonuclear background plasma is considered as a stationary, axially symmetric magnetofluid with radial structure. The column is assumed to be sufficiently long that end effects can be neglected. A multi-group technique has been utilized to examine the α-particles as a collection of particles distributed among a continuous spectrum of confined orbits.

This methodology should apply to any geometry for which the α-particle orbits can be described in terms of adiabatic invariants which change for a single particle very slowly compared to the orbital period. It provides a single mathematical formulation which applies equally well to the initial transient α-dynamics phase immediately following plasma ignition and to the quasi-steady-state phase following. The transition between the two occurs after several slowing-down times, as determined by the plasma core in the fast solenoid case and the surrounding cold plasma blanket in the steady solenoid case.

In the fast solenoid case, the quasi-steady phase is characterized by an α energy spectrum which is stationary and approximately inversely proportional to dE/dt for superthermal α-particles. with the thermal population increasing at the rate of birth of new α particles. The radial profile of the fluid variables is stationary in the interior of the plasma, with particles accumulating near the plasma surface as they slow down. In the steady solenoid case, the α spectrum is similar during the quasi-steady phase, with equilibrium achieved much earlier because of the rapid energy transfer in the colder, denser plasma. In this case, however, steady state is characterized by stationary fluid profiles outside the hot plasma, and particles drift to the hotter plasma as they slow down. Most of the α energy is transferred to the blanket plasma when it is present, which should result in much less heating of the thermonuclear plasma.

In addition to providing a technique for determining the evolution of the α-particle distribution function, the methodology developed here also permits transformation to fluidlike variables for the α-particles through appropriate weighting of moments of the distribution function. The use of a distribution function in an adiabatic-invariant representation results in an enormous increase in the time scale which can be treated by computational techniques—from the cyclotron period range to the slowing-down time range. This enables analysis of the entire duty cycle of a laser solenoid plasma in very reasonable computation times.

REFERENCES

1. D. J. ROSE AND M. CLARK, JR., "Plasmas and Controlled Fusion," MIT Press, Cambridge, Mass., 1961.
2. D. J. SIGMAR AND G. JOYCE, *Nucl. Fusion* **11** (1971), 447.
3. R. D. HAZELTINE, A. A. WARE, D. J. SIGMAR, S. P. HIRSHMAN, J. E. MCCUNE, E. C. CRUME, J. T. HOGAN, AND J. F. CLARKE, *in* "Plasma Physics and Controlled Nuclear Fusion Research 1974" (Proc. 5th Conf. Tokyo, 1974), Intern. At. Energy Agency, Vienna, 1975.
4. A. NOCENTINI, M. TESSAROTTO, AND F. ENGELMANN, *Nucl. Fusion* **15** (1975), 359.
5. D. F. DÜCHS AND D. PFIRSCH, *in* "Plasma Physics and Controlled Nuclear Fusion Research 1974" (Proc. 5th Conf. Tokyo, 1974), Intern. At. Energy Agency, Vienna, 1975.
6. H. TSUJI, M. KATSURAI, T. SEKIGUCHI, AND N. NAKANO, *Nucl. Fusion* **16** (1976), 287.
7. E. G. CORMAN, W. E. LOEWE, G. E. COOPER, AND A. M. WINSLOW, *Nucl. Fusion* **15** (1975), 377.
8. T. E. STRINGER, *Plasma Phys.* **16** (1974), 651.
9. D. G. MCALEES, Report ORNL-TM-4661, 1974.
10. T. W. PETRIE AND G. H. MILEY, *Nucl. Sci. Eng.* **64** (1977), 151.
11. L. C. STEINHAUER, Report MSNW-75-134-2, 1975.
12. H. J. WILLENBERG, Ph.D. thesis, University of Washington, Seattle, 1976.

13. H. J. WILLENBERG, *Nucl. Fusion* **19** (1979), 313.
14. N. A. KRALL AND A. W. TRIVELPIECE, "Principles of Plasma Physics," McGraw-Hill, San Francisco, 1973.
15. H. LEWIS, *in* "Methods in Computational Physics" B. Adler, S. Fernbach, and J. Rottenberg, Eds.), Vol. 9, Academic Press, New York, 1970.
16. D. J. ROSE, *Nucl. Fusion* **9** (1969), 183.
17. T. KAMMASH AND D. L. GALBRAITH, *Nucl. Fusion* **13** (1973), 133.
18. A. A. HUSSEINY AND H. K. FORSEN, *Phys. Rev. A* **1** (1970), 1483.
19. L. SPITZER, JR., "Physics of Fully Ionized Gases," Interscience, New York, 1956.
20. J. M. DAWSON, A. HERTZBERG, R. E. KIDDER, G. C. VLASES, H. G. AHLSTROM, AND L. C. STEINHAUER, *in* "Plasma Physics and Controlled Thermonuclear Research" (Proc. 4th Conf. Madison), Intern. At. Energy Agency, Vienna, 1971.
21. D. J. SIGMAR AND H. C. CHAN, Report ORNL/TM-6027, 1978.
22. V. S. BELIKOV, Y. I. KOLESNICHENKO, AND V. N. ORAEVSKY, *Nucl. Fusion Suppl.* (1972), 301.
23. V. N. TSYTOVICH, *Annu. Rev. Astron. Astrophys.* **11** (1973), 363.